Characterizing graph-nonedge pairs with single-interval Cayley configuration spaces in 3-dimensions

William Sims *   Meera Sitharam*,†

Abstract. We completely characterize graphs G and their non-edges f such that across all 3-dimensional Euclidean realizations of a given G-linkage, f attains lengths in a single interval, answering a question posed by Sitharam and Gao in 2010. More precisely, given any assignment of (Euclidean) edge-lengths for G, f attains a single interval of length values across all assignment of points in $\mathbb{R}^3$ to the vertices of G such that their pairwise distances agree with the given edge lengths. The class is not minor closed, has no obvious well quasi-ordering, and there are infinitely many minimal graphs w.r.t. edge contractions in the complement class. Our characterization overcomes these obstacles, is based on the 2 forbidden minors of the minor-closed class of 3-flattenable graphs, and contributes to the theory of Cayley configurations with applications in analyzing a variety of distance-constrained configuration spaces.

1 Introduction

A *linkage* (G, ℓ) is a pair containing a graph G and a squared Euclidean edge-length map ℓ from the edge set of G to the real numbers. In this paper, we assume that G is a simple, connected graph. A d-dimensional Euclidean *realization* of (G, ℓ) is a map p from the vertex set of G to points in $\mathbb{R}^d$ such that, for each edge uv of G, the Euclidean distance between $p(u)$ and $p(v)$ is $\ell(uv)^{1/2}$. The *d-dimensional configuration space* $\mathcal{C}^d(G, \ell)$ of (G, ℓ) is the set of all d-dimensional realizations of this linkage.

The problem of representing and analyzing the set $\mathcal{C}^d(G, \ell)$, assuming it is non-empty, has applications in computer-aided design, robotics, and molecular science. Graph rigidity theory attempts to determine the dimension of $\mathcal{C}^d(G, \ell)$, for almost all choices of the map ℓ, using only the combinatorics of the graph G. When $d = 2$, there is a well-known combinatorial characterization of graphs G such that $\mathcal{C}^2(G, \ell)$ is finite-dimensional for almost all choices of ℓ. No such result is known for any $d > 2$. See [5], [6], and [12] for an overview of graph rigidity theory and the configuration space problem.

A recent method [11] to represent the set $\mathcal{C}^d(G, \ell)$ is to first choose a set F of nonedges of G - i.e., pairs of vertices that are not edges - and map each point p in $\mathcal{C}^d(G, \ell)$ to the point in $\mathbb{R}^{|F|}$ corresponding to the squared lengths attained by each nonedge in F under p. Let $\phi^d_F(G, \ell)$ denote the image of this map, which is called the *Cayley configuration space of* (G, ℓ) *over* F *in* $\mathbb{R}^d$. Given a point in $\phi^d_F(G, \ell)$ the pre-image of the map gives points in $\mathcal{C}^d(G, \ell)$.

There are several known partial characterizations of graphs G and choices of sets F of nonedges such that the following properties hold. **Property (i)** - $\phi^d_F(G, \ell)$ is convex for any choice of the map ℓ; and **Property (ii)** - the preimage of each point in $\phi^d_F(G, \ell)$ is a finite set [11, 14, 15]. Property (i), which is the focus of this paper, allows for e.g. membership determination, sampling and other computations on $\phi^d_F(G, \ell)$ - by using its convexity and its boundary description - and in deterministic polynomial time for $d \leq 3$ [11]. Note that the d-dimensional configuration space $\mathcal{C}^d(G, \ell)$ could have dimension much larger or smaller than d, and merely bounds the dimension of the image $\phi^d_F(G, \ell)$ from above. Clearly, the latter dimension is at most $|F|$, and generically equals $|F|$ provided F is independent and not implied by G in the sense of graph rigidity [6, 12]. Property (ii), along with some additional geometric information [7, 8, 13–15, 17, 19], enables the pre-image of the map above to be computed in polynomial time for $d \leq 3$.

For $d = 2$, there is a complete characterization of pairs (G, F) satisfying Property (i) [11]. We present a complete characterization for $d = 3$, for the case where F contains exactly one edge.

1.1 Previous work

We formalize Property (i) as follows.

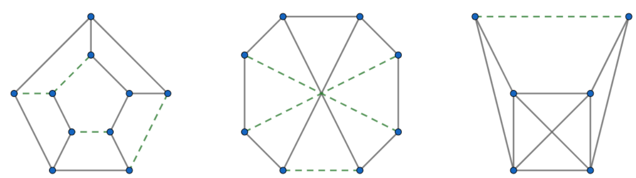

Figure 1: Shown are three pairs (G, F), where F is the set of green nonedges of the graph G. Each pair has the d-convexity property, for any $d \geq 3$.

Definition 1 *Let G be a graph, let F be a set of its nonedges, and let d be any positive integer. The pair (G, F) has the d-*convexity property *if the Cayley configuration space $\phi^d_F(G, \ell)$ is a convex set, for any squared edge-length map ℓ. If (G, F) has this property and F contains exactly one nonedge f, we say that (G, f) has the d-single interval property (d-SIP), since this implies that $\phi^d_f(G, \ell)$ is a single interval.*

For example, each pair in Figure 1 (nonedge sets

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†Affiliate in Department of Mathematics

are shown in green) has the d-convexity property, for any $d \geq 3$. Since the nonedge set in the right-most pair has size 1, this pair has the d-SIP, for any $d \geq 3$. Intuitively, each endpoint of this nonedge traces out a circle in 3-dimensions. In Figure 2, the pair (G, F) on the top left (resp. bottom left) has the d-convexity property, for any $d \geq 3$ (resp. $d \geq 2$). The top left pair does not have 2-convexity. See Theorem 1. The top-right image in this figure is the set $\phi_F^3(G, \ell)$ for the top-left linkage. The bottom-right image in this figure is the set $\phi_F^2(G, \ell)$ for the bottom-left linkage.

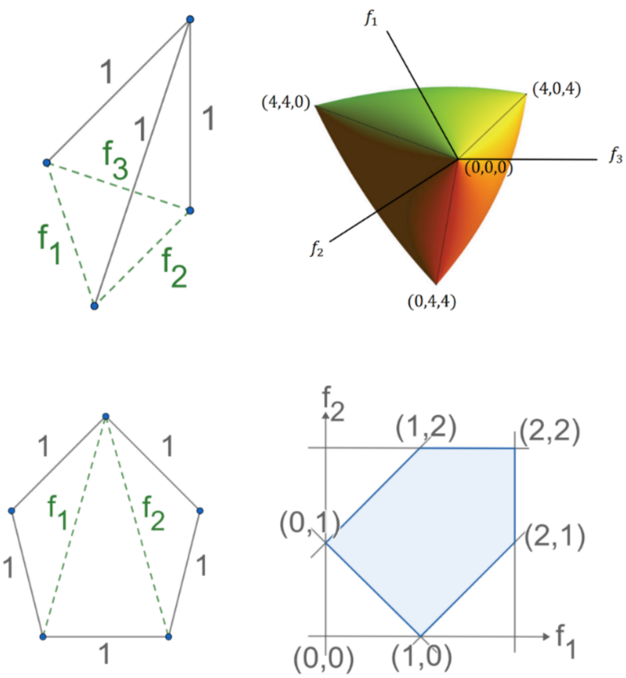

Figure 2: Left: linkages (G, ℓ) and sets F of dashed nonedges. Top-right: the set $\phi_F^3(G, \ell)$ for the top-left linkage; credit to [18]. Bottom-right: the set $\phi_F^2(G, \ell)$ for the bottom-left linkage. Credit for both bottom figures to [11].

Using properties of the Euclidean distance cone, discovered by Schoenberg [10], a strong connection between d-convexity and the property of graph *flattenability* was demonstrated for $d \leq 3$ [11]. A graph G is d-*flattenable* if any linkage (G, ℓ) that has a realization in some dimension also has a d-dimensional realization. For example, consider each pair (G, F) in Figure 1 and the graph $G \cup F$. For the first two pairs, $G \cup F$ is 3-flattenable. For the third pair, $G \cup F$ is 4-flattenable, but not 3-flattenable.

The significance of this connection is that flattenability is a minor-closed property of graphs [4]. Hence, the famous Graph Minor theorem [9] ensures the existence of a finite set $\mathcal{M}_d$ of graphs, called *forbidden minors*, such that the class of d-flattenable graphs is exactly the class of graphs that do not have a minor in $\mathcal{M}_d$. From folkore, the forbidden minors for 1 and 2-flattenability are K_3 and K_4, respectively. The forbidden minors for 3-flattenability are K_5 and $K_{2,2,2}$ [3, 4]. The complete set of forbidden minors for d-flattenability is unknown for any $d > 3$. Clearly, if a graph is d-flattenable, it is also d'-flattenable for any $d' \geq d$

Inspired by the work of Ball [2], the connection between d-flattenability and d-convexity and was generalized to any dimension, and to other norms [16].

Theorem 1 ([16]) *For any graph G, the following statements are equivalent:*

1. *G is d-flattenable.*
2. *For any $d' \geq d$, any subset F of the edges of G and any linkage $(G \setminus F, \ell)$, the Cayley configuration space $\phi_F^{d'}(G \setminus F, \ell)$ is convex.*

This theorem explains why the first two pairs (G, F) in Figure 1 have the d-convexity property, for any $d \geq 3$: as mentioned above, the graphs $G \cup F$ are 3-flattenable. Similarly, for the top left pair (G, F) in Figure 2, the graph $G \cup F$ is 3-flattenable (no K_5 or $K_{2,2,2}$ minor) but not 2-flattenable (K_4 minor). For the bottom left $G \cup F$ is 2-flattenable (no K_4 minor). Hence, this theorem states that the top left (resp. bottom left) (G, F) has the d-convexity property, for any $d \geq 3$ (resp. $d \geq 2$).

These partial characterizations of convexity have been heavily used in the opensource software (EASAL [7][8] and CayMos [20] [13]) for respectively molecular and particle assembly modeling and kinematic mechanism analysis and design and have led to several improvements in those areas, besides efficient algorithms for the core problem of distance constraint graph realization [1].

These are only *partial* characterizations of d-convexity even for $d \leq 3$ because there are many graphs G that are not d-flattenable, and yet there may be choices of sets F of nonedges of G such that (G, F) has the d-convexity property. A complete characterization of these pairs was given for $d = 2$ [11]. We require the following definition to state the theorem.

Definition 2 *A* clique-separator *C of a graph G is a subgraph of G that is a clique whose vertex set separates G. Let H_i be a connected component of $G \setminus V(C)$ and let G_i be the subgraph of G induced by $V(H_i) \cup V(C)$. The graph G_i is a C*-clique-sum component of *G and G is the C-clique-sum of the set of these components.*

For any positive integer k, a minimal k-clique-sum decomposition of *G is a tree such that the root node is G and each internal node G' is the C-clique-sum of its children, for some clique-separator C of G' on at most k vertices. A* minimal k-clique-sum component of *G is any leaf node in any such decomposition. A* minimal k-clique-sum graph *is a graph that has no clique-separator on at most k vertices.*

For example, the graph on the right in Figure 1 contains two 2-clique-separators, and its minimal 2-clique-sum components are the K_4 and the two K_3 subgraphs. On the other hand, the entire graph is a minimal 1-clique-sum graph. It is straightforward to verify that every minimal k-clique-sum decomposition of a graph has the same set of leaf nodes.

Theorem 2 ([11]) *Let G be a graph and let F be a set of its nonedges. Then, the following statements are equivalent:*

1. *(G, F) has the 2-convexity property.*
2. *For every minimal 2-clique-sum component H of $G \cup F$ that contains any subset of F, H has no K_4 minor.*

Algorithmically speaking, note that Statement (2) of this theorem is equivalent to H being 2-flattenable. For a fixed, finite set of minors, there is a polynomial-time algorithm to determine if a graph has a minor in this set [9]. Hence, we can efficiently check if a pair (G, F) satisfies Statement (2) in Theorem 2. A key ingredient in the proof of this theorem is the following complete characterization of pairs with the 2-SIP.

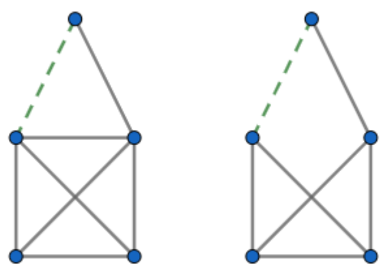

Figure 3: The pair on the left has the 2-SIP, while the pair on the right does not. This illustrates that d-convexity is not a minor-closed property.

Theorem 3 ([11]) *Let G be a graph and let f be one of its nonedges. Then, the following statements are equivalent:*

1. *(G, f) has the 2-SIP.*
2. *For every minimal 2-clique-sum component H of $G \cup \{f\}$ that contains f, H has no K_4 minor.*

There are two major obstacles to general d-dimensional extensions of "lucky" results such as Theorems 2 and 3 that turn out to be essentially finite forbidden minor characterizations with the associated efficient algorithms. First, d-convexity is not a property of graphs, but of pairs containing a graph and a set of its nonedges. Second, d-convexity is not a minor-closed property. For example, the pair on the left in Figure 3 has the 2-SIP, but the pair on the right does not. Hence, the Graph Minor theorem [9] does not directly apply, and it is not clear whether these properties have anything akin to finite forbidden minor characterizations at all.

2 Results

The following is our main theorem.

Theorem 4 *Let G be a graph and let f be one of its nonedges. Then, the following statements are equivalent:*

1. *(G, f) has the 3-SIP.*
2. *For every minimal 3-clique-sum component H of $G \cup \{f\}$ that contains f, f is contracted in every K_5 and $K_{2,2,2}$ minor of H.*

Before we discuss the main ideas in the proof of this theorem, it is instructive to see why a simpler generalization of Theorem 3 fails for $d = 3$. For instance, consider this alternative proposal for Statement (2) in Theorem 4: every minimal 3-clique-sum component of $G \cup \{f\}$ that contains f has no K_5 or $K_{2,2,2}$ minor. Observe that for the pair (G, f) on the right in Figure 1, the graph $G \cup \{f\}$ is a minimal 3-clique-sum graph that has a K_5 minor. Hence, the alternative proposed statement indicates that (G, f) should not have the 3-SIP. However, this pair does have the 3-SIP, which is accurately indicated by Statement (2) in Theorem 4. In particular, Theorem 4 correctly recognizes that although $G \cup \{f\}$ has a K_5 minor, f must be contracted to reach it.

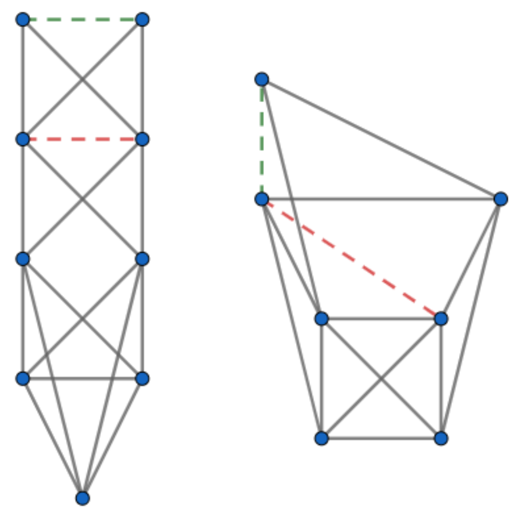

Figure 4: Minimal pairs in the proof of the converse of Theorem 4.

Finally, we briefly summarize the proof of Theorem 4. The proof of the forward direction is by contrapositive. First, we assume that there exists a minimal 3-clique-sum component H of $G \cup \{f\}$ that contains f and has a K_5 and $K_{2,2,2}$ minor in which f is not contracted. Then, we show that this implies that (G, f) does not have the 3-SIP.

Along the way to the implication above, we reduce the problem by finding a pair (G', f) such that G' is smaller than G and it is sufficient to show that (G', f) does not have the 3-SIP. The graph G' is either a clique-sum component of G that contains H or a minor of G obtained via a single contraction on one of its edges. Eventually, we reach a minimal pair that cannot be further reduced without forcing the resulting pair to have the 3-SIP.

The remainder of the proof takes advantage of the special properties of such minimal pairs. Examples of such minimal pairs (G, f) are shown in Figure 4, where f is the green nonedge. For each pair, observe that $G \cup \{f\}$ is a minimal 3-clique-sum graph and any contraction on an edge of G does one of the following:

(i) removes every K_5 and $K_{2,2,2}$ minor in which f is not contracted,

(ii) separates f from any K_5 and $K_{2,2,2}$ minor in which it is not contracted via some clique-separator, or

(iii) neither (i) nor (ii) and maps f to another edge of G.

Another minimal pair (G_2, f_2) can be obtained from the pair (G_1, f_1) on the left as follows. The graph G_2 is obtained from G_1 by copying the subgraph "between" the green and red nonedges and gluing the red nonedge in this copy to the green nonedge in G_1. The nonedge f_2 is the green nonedge of G_2 that is not contained in G_1. Repeating this process yields an infinite family of minimal pairs, none of which has the 3-SIP. The set of minimal pairs being infinite is one of the main obstacles in characterizing pairs with the d-SIP caused by the fact that the d-SIP is not a minor-closed property.

Returning to the proof sketch, wlog, we can assume that (G, f) is a minimal pair. The case where no contraction on an edge of G satisfies (ii) is easily handled. Otherwise, we show that there exists another nonedge f' of G (the red nonedges in Figure 4) such that (G, f') does not have the 3-SIP. Furthermore, the subgraph "between" f and f' is a special case of 3-flattenable graph, called a partial 3-tree. These two statements are then sufficient to show that (G, f) does not have the 3-SIP.

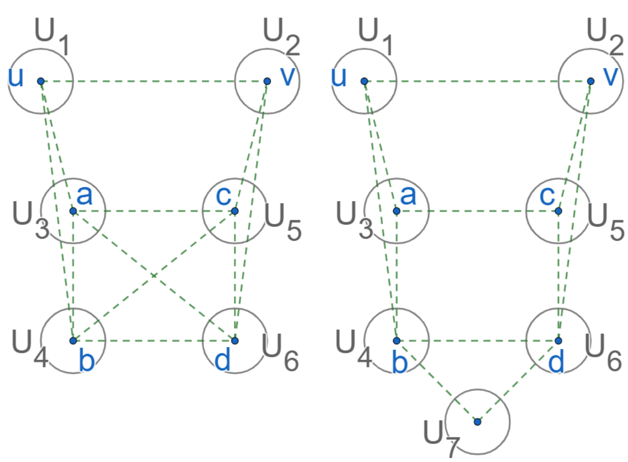

Figure 5: Quotient graphs used in the proof of the forward direction of Theorem 4.

Next we sketch the proof of the converse of the theorem, which is simpler. It uses the following independently interesting lemma.

Lemma 1 *Let G be a graph and let f_1 and f_2 be distinct nonedges of G. If (G, f_2) and $(G \cup \{f_2\}, f_1)$ both have the d-SIP, then (G, f_1) has the d-SIP.*

For example, consider the pair (G, f_1) on the right in Figure 3. Let f_2 be the nonedge such that $G \cup \{f_2\}$ is the graph on the left. Since $G \cup \{f_2\}$ is 3-flattenable, (G, f_2) has the 3-SIP, by Theorem 1. Also, as discussed above, $(G \cup \{f_2\}, f_1)$ has the 2-SIP, which implies that it has the 3-SIP. Thus, Lemma 1 says that (G, f_1) has the 3-SIP.

The remainder of the proof proceeds as follows. Start by assuming Statement (2) in Theorem 4. The strength of this statement greatly restricts the class of graphs to inspect. In particular, it allows us to partition the vertex set of $G \cup \{f\}$ such that the quotient graph is one of the graphs in Figure 5 and the endpoints of f, say u and v, are contained in the sets U_1 and U_2, respectively. Furthermore, we can identify vertices $a \in U_3$, $b \in U_4$, $c \in U_5$, and $d \in U_6$ that play a significant role in all paths between the endpoints of f. Next, we consider the set of pairs

$$F = \{ua, ub, vc, cd, ab, ac, ad, bc, bd, cd\}$$

(see Figure 5) and show that for any $f' \in F$, if f' is a nonedge, then (G, f') has the 3-SIP. Finally, we use these facts in combination with Lemma 1 applied to the nonedges in F to show that (G, f) has the 3-SIP.

3 Open Problems

The first problem is to given an efficient algorithm to determine whether a pair (G, f) satisfies the characterization of this paper, Theorem 4. Since it is not directly a finite forbidden minor characterization, the polynomial time algorithm of [9] cannot be used. For reasons mentioned earlier, the algorithm for 2-SIP cannot be used. The second problem is to prove or refute the following conjecture.

Conjecture: Let G be a graph and let F be a set of its nonedges. Then, the following statements are equivalent:

1. (G, F) has the d-convexity property.
2. For each nonedge $f \in F$ and every minimal d-clique-sum component H of $G \cup F$ that contains f, f is contracted in every d-flattenability forbidden minor of H.

Note that the number of forbidden minors for d-flattenability is conjectured to grow quickly with d [4]. Hence, a proof of this conjecture should avoid using properties of the explicit d-flattenability forbidden minors. The previous proof of Theorem 3 in [11] and the proof of Theorem 4 in this paper strongly rely on knowledge of the explicit minors for d-flattenability for $d \leq 3$.

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