Conditional Lower Bounds for Dynamic Geometric Measure Problems

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1 Introduction

In 1995, Gajentaan and Overmars [7] introduced the notion of 3SUM hardness, showing that a number of problems in computational geometry can not be solved in subquadratic time, assuming the so-called 3SUM problem can not be solved in subquadratic time.[1] The general approach of proving polynomial lower bounds based on a few conjectures about key problems has since grown into its own subfield of complexity theory known as *fine-grained complexity*. The most popular of these conjectures concern the aforementioned 3SUM problem, All-Pairs-Shortest-Paths (APSP), Boolean Matrix Multiplication (BMM), Triangle finding in a graph, Boolean Satisfiability (SAT) and the Orthogonal Vectors problem (2OV) (see for example the introductory surveys by Bringmann [1] and V. V. Williams [13]).

Pătraşcu [12] launched the study of such polynomial lower bounds for dynamic problems, where instead of simply computing a function on a single input, we want to be able to update that input and get the corresponding output of the function without having to recompute it from scratch. In particular, he introduced the Multiphase problem and showed a polynomial lower bound on its complexity, conditioned on the hardness of the 3SUM problem. Using the Multiphase problem as a stepping stone, he showed conditional hardness results for a variety of dynamic problems. Improvements and other conditional lower bounds for dynamic problems (data structure problems) have since appeared in the literature. Of particular interest for the purpose of this work is a paper by Kopelowitz et al. [11] where the approach of Pătraşcu is improved by showing a tighter reduction from 3SUM to the so-called Set Disjointness problem (an intermediate problem between 3SUM and the Multiphase problem), as well as a paper by V. V. Williams and Xu [14], which obtains a similar reduction from the so-called Exact Triangle problem. Also particularly relevant here is the work of Henzinger et al. [9], who show that many of the known bounds on dynamic problems can be derived (and even strengthened) by basing proofs on a hardness conjecture about the Online Boolean Matrix-Vector Multiplication (OMv) problem which they introduce.

While computational geometry was one of first fields where conditional lower bounds for algorithms were applied, for example by showing that determining if a point set is in general position is 3SUM hard [7], the progress in conditional lower bounds for dynamic problems has not found widespread application to computational geometry; recent work has been largely confined to improved upper bounds. In this work, we exploit the results of Pătraşcu, Kopelowitz et al., V. V. Williams and Xu, and Henzinger et al. to give conditional polynomial lower bounds for a variety of dynamic problems in computational geometry, based on the hardness of 3SUM, APSP and Online Boolean Matrix-Vector Multiplication.

[1] In 2014, Grønlund and Pettie [8] showed that the 3SUM problem can be solved in (slightly) subquadratic time. The modern formulation thus replaces "subquadratic" with "truly subquadratic", i.e. $O(n^{2-\varepsilon})$ for some constant $\varepsilon > 0$.

Almost all the problems we study here share the common characteristic of being about computing a single global metric for a set of objects in space subject to updates. Moreover, in the static case (where there are no updates) most of these metrics can be computed in worst-case $O(n \log n)$ time using standard computational geometry results. In particular, we show conditional hardness results for orthogonal range marking, maintaining the number of maximal or extremal points in a set of points in $\mathbb{R}^3$, dynamic approximate square set cover (answering a question by Chan et al. [5]), problems related to Klee's Measure Problem, problems related to finding the largest empty disk in a set of points, testing whether a set of disks covers a given rectangle, and querying for the size of the i'th convex layer of a set of points in the plane. We also give an unconditional lower bound for the incremental Hypervolume Indicator problem in $\mathbb{R}^3$, where the goal is to maintain the volume of the union of a set of axis-aligned boxes which all have the origin as one of their vertices.

1.1 Setting and computational model

We work in the standard Word RAM model, with words of $w = \Theta(\log n)$ bits unless otherwise stated, and for randomized algorithms we assume access to a perfect source of randomness. We base our conditional lower bounds on the following well known hardness conjectures.

▶ **Conjecture 1** (3SUM conjecture). *The following problem (3SUM) requires $n^{2-o(1)}$ expected time to solve: given a set of n integers, decide if three of them sum up to 0.*

▶ **Conjecture 2** (APSP conjecture). *The following problem (APSP) requires $n^{3-o(1)}$ expected time to solve: given an integer-weighted directed graph G on n vertices with no negative cycles, compute the distance between every pair of vertices in G.*

In addition to being the basis for these standard conjectures in fine-grained complexity, the 3SUM problem and the APSP problem are related in other ways (see [14]). In particular, they both fine-grained reduce to the Exact Triangle problem, meaning that if either the 3SUM conjecture or the APSP conjecture is true, then the following conjecture is true.

▶ **Conjecture 3** (Exact Triangle conjecture). *The following problem (Exact Triangle) requires $n^{3-o(1)}$ expected time to solve: given an integer-weighted graph G and a target weight T, determine if there is a triangle in G whose edge weights sum to T.*

We also consider a conjecture introduced by Henzinger et al. [9], which can be thought of as a weakening of the informal conjecture which says that "combinatorial" matrix multiplication on $n \times n$ matrices requires essentially cubic time (note that the term "combinatorial" is not well defined).

▶ **Conjecture 4** (OMv conjecture). *The following problem (OMv) requires $n^{3-o(1)}$ expected time to solve:*
We are given a $n \times n$ boolean matrix M. We can preprocess this matrix, after which we are given a sequence of n boolean column-vectors of size n denoted by $v_1, ..., v_n$, one by one. After seeing each vector v_i, we must output the product Mv_i before seeing v_{i+1}.

1.2 Main results

In the full version [6] of this paper we obtain (conditional) polynomial lower bounds for a variety of dynamic geometric problems, and an unconditional bound for the incremental Hypervolume Indicator problem in $\mathbb{R}^3$. Our bounds are stated as inequalities which imply trade-offs between achievable update and query times. The lower bounds we get on the

<table>
<tr><th>Problem</th><th>Upper Bound</th><th>Lower Bound</th></tr>
<tr><td>Square Range Marking</td><td>$\tilde{O}(n^{1/2})$ †,‡ [2]</td><td>From Exact Triangle:
$n^{1/4-o(1)}$ †

From OMv:
$n^{1/2-o(1)}$ †</td></tr>
<tr><td>Counting Extremal Points in $\mathbb{R}^3$</td><td rowspan="4">$O^*(n^{7/8})$ † [3]

$O^*(n^{11/12})$ ‡ [4]</td><td rowspan="6">From Exact Triangle:
$n^{1/5-o(1)}$ †,‡
$n^{1/4-o(1)}$ ‡,$

From OMv:
$n^{1/2-o(1)}$ †,‡</td></tr>
<tr><td>Largest Empty Disk in Query Region</td></tr>
<tr><td>Largest Empty Disk in a Set of Disks</td></tr>
<tr><td>Rectangle Covering with Disks</td></tr>
<tr><td>Square Covering with Squares</td><td>$\tilde{O}(n^{1/2})$ ‡ [15]</td></tr>
<tr><td>Convex Layer Size in $\mathbb{R}^2$</td><td></td></tr>
<tr><td>Counting Maximal Points in $\mathbb{R}^3$</td><td>$\tilde{O}(n^{2/3})$ ‡ [4]</td><td rowspan="4">From Exact Triangle:
$n^{1/4-o(1)}$ †
$n^{1/3-o(1)}$ ‡

From OMv:
$n^{1/2-o(1)}$ †,‡</td></tr>
<tr><td>$O(n^\alpha)$-approx. Weighted Square Set Cover</td><td></td></tr>
<tr><td>Klee's Measure Problem with Squares</td><td>$\tilde{O}(n^{1/2})$ ‡ [15]</td></tr>
<tr><td>Discrete KMP with Squares</td><td>$O(n^{1/2})$ †,‡ [16]</td></tr>
<tr><td>Depth Problem with Squares</td><td>$\tilde{O}(n^{1/2})$ ‡ [15]</td><td>From Exact Triangle:
$n^{1/3-o(1)}$ †,‡

From OMv:
$n^{1/2-o(1)}$ †,‡</td></tr>
<tr><td>$O(1)$-approximate Square Set Cover</td><td>$O^*(n^{1/2})$ ‡ [5]</td><td>From OMv:
$n^{1/3-o(1)}$ †,‡</td></tr>
<tr><td>Hypervolume Indicator in $\mathbb{R}^3$</td><td>$\tilde{O}(n^{2/3})$ ‡ [4]</td><td>$\Omega(\sqrt{n})$ #</td></tr>
</table>

† per-operation runtime in the incremental setting.
‡ amortized runtime in the fully-dynamic setting.
$ assuming $n^{1+o(1)}$ expected preprocessing time.
# unconditional lower bound in the incremental setting on amortized time, assuming at most polynomial time preprocessing, or on worst-case time without preprocessing assumptions.

Table 1 Non-trivial known upper bounds and new (at the time of the first version of this paper being made public) lower bounds on the maximum over update and query time derived from the Exact Triangle conjecture, the OMv conjecture or (in the case of the Hypervolume Indicator problem) unconditionally. The $\tilde{O}$ notation hides polylog factors, while the O^* notation hides factors which are $o(n^\varepsilon)$ for an arbitrarily small constant $\varepsilon > 0$. All upper bounds are for data structures with at most $O^*(n)$ preprocessing. Note that the lower bounds for Square Range Marking also hold in the case of a static set of points (with some assumptions on preprocessing time) and that the lower bound for the Depth Problem derived from the OMv conjecture also holds for amortized runtime in the incremental setting. The lower bound obtained for counting maximal points has since been superseded by the more general result of Jin and Xu [10] who obtain lower bounds also in higher dimension.

maximum of both are summarized in Table 1, together with known upper bounds. Note that the bounds we get for squares or square ranges imply the same bounds for rectangles or general orthogonal ranges, although we sometimes get better trade-offs in these cases.

Some of the lower bounds reveal interesting separations between geometric dynamic problems whose operations can be supported in subpolynomial or $O(n^{\varepsilon})$ time and similar problems which require polynomial time with a fixed exponent (under the hardness conjectures we consider).

- Orthogonal range queries with dynamic updates on single points can be done with polylog time operations, while dynamic updates on orthogonal ranges of points require polynomial time.
- Dynamically maintaining maximal points in a point set can be done in polylog time in $\mathbb{R}^2$, while maintaining only their number in $\mathbb{R}^3$ already requires polynomial time.
- The same separation between dimensions 2 and 3 applies for maintaining (the number of) extremal points.
- Related to the previous point, the ability to query for the size of any convex layer on a dynamic set of points in $\mathbb{R}^2$ requires polynomial time (compared to polylog time when we are only interested in the first convex layer, i.e. the convex hull).
- Maintaining a $O(1)$-approximation for the size of dynamic unit square set cover can be done in $2^{O(\sqrt{\log n})}$ amortized time per update [5], while maintaining the size of a $O(n^{\alpha})$-approximation (for a constant $0 \leq \alpha < 1$) requires polynomial time for arbitrarily sized squares (with an exponent dependent on α).
- In the weighted case of the previous problem, we also get such a separation: $O(1)$-approximate weighted unit square set cover can be done in $O(n^{\varepsilon})$ time [5] while $O(n^{\alpha})$-approximate weighted dynamic square set cover requires polynomial time, with an exponent independent of α.

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